Published by Institute of Physics Publishing for SISSA

Received: *January 04, 2007*
Accepted: *January 30, 2007*
Published: *March 1, 2007*

$1/2$-BPS correlators as $c = 1$ S-matrix

Antal Jevicki

Department of Physics, Brown University
Providence, RI 02912, U.S.A.
E-mail: antal@het.brown.edu

Tamiaki Yoneya

Institute of Physics, University of Tokyo
Komaba, Meguro-ku, Tokyo 153-8902, Japan
E-mail: tam@hep1.c.u-tokyo.ac.jp

Abstract: We argue from two complementary viewpoints of Holography that the 2-point correlation functions of 1/2-BPS multi-trace operators in the large-N (planar) limit are nothing but the (Wick-rotated) S-matrix elements of $c = 1$ matrix model. On the bulk side, we consider an Euclideanized version of the so-called bubbling geometries and show that the corresponding droplets reach the conformal boundary. Then the scattering matrix of fluctuations of the droplets gives directly the two-point correlators through the GKPW prescription. On the Yang-Mills side, we show that the two-point correlators of holomorphic and anti-holomorphic operators are essentially equivalent with the transformation functions between asymptotic *in*- and *out*-states of $c = 1$ matrix model. Extension to non-planar case is also discussed.

Keywords: AdS-CFT Correspondence, Matrix Models, Gauge-gravity correspondence.

Contents

1. Introduction **1**

2. Euclideanized bubbling geometries **2**

3. Scattering of droplets **6**
3.1 Boundary S-matrix 6
3.2 Hamiltonian formalism 9

4. Comparison with the two-point functions of multi-trace operators **10**

5. Droplet scattering from the viewpoint of the $c = 1$ matrix model **14**
5.1 $c = 1$ scattering interpretation of the correlators 14
5.2 Case of higher-genus 16

6. Conclusion **20**

A. Genus expansion from exact formulas **21**

1. Introduction

A certain class of correlation functions of $\mathcal{N} = 4$ super Yang-Mills theory can be computed exactly using a free-field approximation. A very interesting nontrivial example of such cases is the two-point correlator of multi-trace local operators belonging to the 1/2-BPS sector. In this case, the free-field approximation of the gauge theory can further be mapped to a one-dimensional matrix model [1, 2]. Hence, such correlators can be conveniently classified by using the language of one-dimensional non-relativistic free fermions. It is quite remarkable that the similar fermion-liquid ('droplet') picture also arises in the holographically dual description, namely, on the supergravity side [3] as the classification of a class of the so-called bubbling geometries with the same amount of (super)symmetries as the 1/2-BPS sector of the gauge theory. The correspondence of both sides in this particular sector can be regarded as yet another evidence for the AdS/CFT correspondence.

It would be of some interest to see whether this correspondence of *classification* at the level of spectrum could be further extended to agreements of physical observables. The purpose of the present note is to provide such an example. According to the standard prescription of the AdS/CFT correspondence, the gauge-theory correlators are interpreted as the amplitudes of propagation of supergravity modes in the bulk, connecting conformal boundary to conformal boundary in the AdS background [4]. However, it has not been clear what should be the corresponding interpretation of the above 1/2-BPS two-point correlators for general multi-trace operators.

We will argue that a reasonable holographic interpretation of two-point correlators naturally emerges if we consider the bubbling geometries in an Euclideanized picture. The two-dimensional droplets characterizing the bubbling geometries are then infinite domains which extend to the conformal boundary of the asymptotically EAdS background.[1] As opposed to a circular droplet which is located close to the center of geometries in the Lorentzian case, the ground-state droplet is an infinite wedge region bounded by a hyperbola. This can be related to the fermi sea of the ground state of the $c = 1$ matrix model. Since the excited states of the droplet propagate along the hyperbola, we can naturally identify the scattering amplitudes of excitation modes along the hyperbola to be the correlators on the gauge theory side through the standard interpretation.

On the other hand, from the viewpoint of the $c = 1$ matrix model, we can argue that the correlators of 1/2-BPS multi-trace operators which take a form of correlation between holomorphic and anti-holomorphic matrix operators is reinterpreted as defining an S-matrix as the amplitudes of transformation between incoming and outgoing multi-particle states of excitations near the fermi sea.

In the next section, we first describe an Euclideanized version of bubbling geometries. Then in section 3, we discuss the scattering amplitudes (which we call 'boundary S-matrix') of excited modes along the ground-state droplet in classical approximation. In section 4, we discuss some examples of matching between boundary S-matrix and the correlators. We also study the limit of large R-charge momentum. The latter will explain the origin of a previous partial proposal due to ref. [6] for a possible relation of the 1/2-BPS correlators with the vertex of collective field theory of the $c = 1$ matrix model. In section 5, we discuss the S-matrix interpretation of 1/2-BPS correlators from the standpoint of the $c = 1$ matrix model *per se*. This allows us to provide an intrinsic interpretation of the correlators as an S-matrix within a logic of matrix model. We also provide a nontrivial example of higher-genus effect which shows the correspondence of 1/2-BPS correlators with the $c = 1$ S-matrix to arbitrary genera in the large momentum limit.

2. Euclideanized bubbling geometries

It is useful to first recall how the ground state, $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry, is embedded in the LLM metric [3] in the Lorentzian case,

$$ds^2 = -h^{-2}(d\tau + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + y e^G d\Omega_3^2 + y e^{-G} d\tilde{\Omega}_3^2. \tag{2.1}$$

All of the functions in this metric (and also the RR-fields which are suppressed here) are determined[2] by specifying the value, either 1/2 or $-1/2$, of a scalar function $z(x_i, y)$ on the plane D at $y = 0$. Under this boundary condition, the Laplace equation for $z(x_i, y)$

$$\partial_i\partial_i z + y\partial_y\left(\frac{\partial_y z}{y}\right) = 0 \tag{2.2}$$

[1] This possibility was first emphasized in [5].

[2] The equations are $h^{-2} = 2y\cosh G, \quad z = \frac{1}{2}\tanh G, \quad y\partial_y V_i = \epsilon_{ij}\partial_j z, \quad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z.$

JHEP03(2007)001

is solved as

$$z(x_1,x_2,y)=\frac{y^2}{\pi}\int_{\mathrm{D}}dx'_1dx'_2\frac{z(x'_1,x'_2,0)}{[(x-x')^2+y^2]^2}. \tag{2.3}$$

The ground state corresponds to a circular droplet of radius $r_0=\sqrt{4\pi g_sN}$

$$z(x_1,x_2,0)=\begin{cases}-1/2\ (x_1,x_2)\in \text{circular disk of radius } r_0\\ +1/2 \text{ otherwise}\end{cases} \tag{2.4}$$

which gives

$$\begin{aligned}z(x_1,x_2,y)-\frac{1}{2}&=-\frac{y^2}{\pi}\int_{\mathrm{Disk}}\frac{r'dr'd\phi'}{r^2+r'^2-2rr'\cos\phi'+y^2]^2}\\ &=\frac{r^2-r_0^2+y^2}{2\sqrt{(r^2+r_0^2+y^2)^2-4r^2r_0^2}}-\frac{1}{2},\end{aligned} \tag{2.5}$$

using spherical coordinates for the 2-dimensional plane $(x_1,x_2)=r(\cos\phi,\sin\phi)$. Then defining new coordinates ρ,θ and $\tilde{\phi}$ by

$$y=r_0\sinh\rho\sin\theta,\quad r=r_0\cosh\rho\cos\theta,\quad \tilde{\phi}=\phi-\tau, \tag{2.6}$$

the LLM metric above reduces to the standard $\mathrm{AdS}_5\times\mathrm{S}^5$ metric expressed in terms of the global coordinate,

$$ds^2=r_0\Big[-\cosh^2\rho d\tau^2+d\rho^2+\sinh^2\rho d\Omega_3^2+d\theta^2+\cos^2\theta d\tilde{\phi}^2+\sin^2\theta d\tilde{\Omega}_3^2\Big]. \tag{2.7}$$

Let us now perform a (double) Wick rotation [7] $\tau\to -i\tau,\phi\to -i\psi$ ($\tilde{\psi}\equiv\psi-\tau\to -i\tilde{\psi}$) under which both the AdS metric and the RR-fields are transformed 'covariantly' into the Euclideanized AdS ($\mathrm{EAdS}_5\times\ \mathrm{S}^{4,1}$) background with the metric,

$$ds^2=r_0\Big[\cosh^2\rho d\tau^2+d\rho^2+\sinh^2\rho d\Omega_3^2+d\theta^2-\cos^2\theta d\tilde{\psi}^2+\sin^2\theta d\tilde{\Omega}_3^2\Big]. \tag{2.8}$$

Since the signature of this metric is still $9+1$ in 10-dimensional sense, supersymmetries can be preserved by a suitable renaming of spinor variables. The two-dimensional coordinates (x_1,x_2) are then transformed as

$$x_1\to x_1=r\cosh\psi,\quad x_2\to ix_2=ir\sinh\psi. \tag{2.9}$$

This exercise implies that for discussing generic Euclideanized LLM ansatz, it is sufficient to make the double Wick rotations $x_2\to ix_2$ and $\tau\to -i\tau$. The vector field V_i must also be rotated covariantly as $V_1\to -iV_1,\ V_2\to V_2$.

Therefore the Laplace equation now becomes a hyperbolic wave equation

$$(\partial_1^2-\partial_2^2)z+y\partial_y\left(\frac{\partial_y z}{y}\right)=0. \tag{2.10}$$

Although it is actually sufficient to consider a wedge region $x_1^2-x_2^2\geq 0$, $x_1\geq 0$ which we hereafter denote by the symbol W, for discussing the asymptotically EAdS backgrounds, let us consider this wave equation in the whole (Wick-rotated) (x_1, x_2) plane which we denote by the same symbol D as in the Lorentzian bubbling geometries. The region outside the W will be denoted by $\overline{\mathrm{W}} \equiv \mathrm{D}-\mathrm{W}$. We have to specify the boundary condition at $y = 0$ for the whole plane D just as in the Lorentzian case by demanding smoothness of solutions at $y = 0$.

The solution must satisfy the boundary condition $z(x_i, y)|_{y=0} = z(x_i, 0) = 1/2$ or $-1/2$ at $y = 0$. Under this boundary condition, the appropriate solution which includes the above EAdS background as the ground state is

$$z(x_1, x_2, y) = -\frac{y^2}{\pi}\mathrm{Im}\Big[\int_{\mathrm{D}} dx_1' dx_2' \frac{z(x_1', x_2', 0)}{[(x_1 - x_1')^2 - (x_2 - x_2')^2 + y^2 - i\epsilon]^2}\Big]. \tag{2.11}$$

This is verified by noting that

$$\lim_{y\to 0}\Big[\frac{y^2}{[(x_1 - x_1')^2 - (x_2 - x_2')^2 + y^2 - i\epsilon]^2} - \frac{y^2}{[(x_1 - x_1')^2 - (x_2 - x_2')^2 + y^2 + i\epsilon]^2}\Big]$$
$$= -2\pi i \delta(x_1 - x_1')\delta(x_2 - x_2'). \tag{2.12}$$

We here chose the boundary condition outside the wedge region W as

$$z(x_1', x_2', 0) = 1/2 \quad \text{for} \quad (x_1', x_2') \in \overline{\mathrm{W}}. \tag{2.13}$$

After obtaining solutions, we can restrict ourselves to the wedge region $(x_1, x_2) \in \mathrm{W}$ for general nonzero y, which is related to the Euclidean plane D of Lorentzian solutions by (2.9).

For example, the EAdS solution discussed above is obtained by replacing the circular disk (with value $-1/2$) of the Lorentzian case by an infinite domain H ($0 < r < r_0$) in the wedge region W bounded by a hyperbola at $r = r_0$. It can be checked by explicitly evaluating the above integral that the expression of z for the EAdS takes the same form as in the Lorentzian case

$$-\frac{y^2}{\pi}\mathrm{Im}\Big[\int_{\mathrm{D}} dx_1' dx_2' \frac{z(x_1', x_2', 0)}{[(x_1 - x_1')^2 - (x_2 - x_2')^2 + y^2 - i\epsilon]^2}\Big]$$
$$= 1/2 + \frac{y^2}{\pi}\mathrm{Im}\Big[\int_{\mathrm{H}} dx_1' dx_2' \frac{1}{[(x_1 - x_1')^2 - (x_2 - x_2')^2 + y^2 - i\epsilon]^2}\Big]$$
$$= \frac{1}{2}\frac{r^2 - r_0^2 + y^2}{\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} \to \begin{cases} 1/2 & \text{if } r > r_0\,,\ y = 0 \\ -1/2 & \text{if } r < r_0\,,\ y = 0 \end{cases} \tag{2.14}$$

as it should be, since the ground state droplet in either metric has no explicit dependence on the angle $\phi \leftrightarrow i\psi$.

Thus we can define Euclideanized bubbling geometries by setting hyperbolical droplet at $y = 0$ in a similar way as in the Lorentzian case. The excited droplets can have arbitrary holes and islands of droplets in the wedge region W. The hyperbolical shape of the ground-state droplet implies that in comparing with the free-fermion picture of matrix models, we should consider the usual $c = 1$ model with inverted harmonic oscillator potential,

JHEP03(2007)001

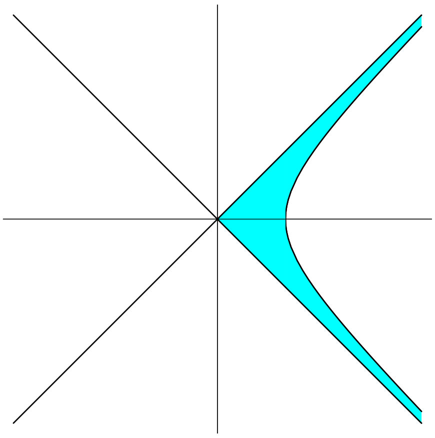

Figure 1: The shaded region (H) in the wedge region W in the plane D represents the ground-state droplet.

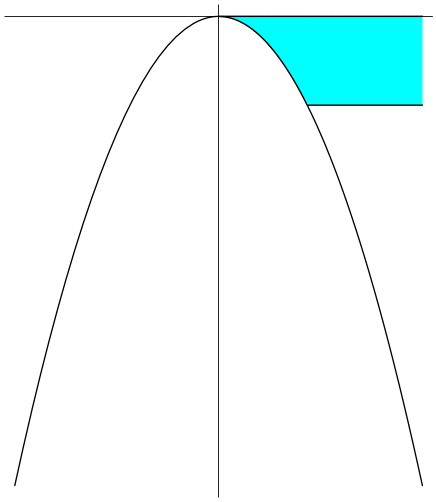

Figure 2: In the fermion picture, the ground state droplet corresponds to filling levels from the top of the potential.

instead of the ordinary harmonic potential of the Lorentzian case. There are two regions (corresponding to positive or negative values of x_1) separated by the inverted potential. But for discussing deformations of the ground-state hyperbola in the classical approximation, it is sufficient to consider only one of them. Since after all we are discussing the classical supergravity solutions which are only justified in the large N limit, we are allowed to treat the fermi sea in the semi-classical approximation on the matrix-model side too.

There is actually a subtle difference that the droplet should be considered only in the wedge region W. In terms of fermionic picture, this amounts to considering one-particle levels only up to the top of the inverted harmonic potential. See Figs.1 and 2. The levels above the potential must be totally ignored, and the ground state is the one in which the levels are filled from the top of potential, *not* from the (infinite) bottom of a suitably regularized potential as usual. In other words, particles and holes are interchanged there, comparing with case of the usual $c = 1$ matrix model. Clearly, however, these differences do not matter again in the semi-classical treatment of droplets.

Now going back to the EAdS geometry obtained by our Wick rotation, the wedge region W of the droplet plane reaches the conformal boundary corresponding to the limit $\psi \to \pm\infty$ which implies $\rho \to \infty$. In discussing the boundary-to-boundary propagation of supergravity modes in the AdS background, it is usually most convenient to use Poincaré coordinates. Therefore let us try to define the corresponding coordinates for general (Euclidean) LLM geometries. Since we will be interested in deformations of the ground-state droplet in the fixed plane W with two SO_4 symmetries kept intact, it is sufficient to deal with four coordinates (τ, x_1, x_2, y). They can be reparametrized as $(\tau, \rho, \psi, \theta)$ by defining

$$x_1 = r\cosh\psi, \quad x_2 = r\sinh\psi, \quad r = r_0\cosh\rho\cos\theta, \quad y = r_0\sinh\rho\sin\theta \tag{2.15}$$

which are the same coordinates as we have used for the EAdS background. Then, for the two-dimensional part described by (τ, ρ), we can make a coordinate transformation to (u, v_4) defined below, where the direction of v_4 is interpreted as the 4-th direction of the conformal boundary, by assuming that we restrict ourselves to the bubbling geometries which reduce to EAdS background asymptotically as $u \to \infty$:

$$\cosh\rho \cosh\tau = \frac{1}{2u}\Big(1 + u^2(1 + v_4^2)\Big), \tag{2.16}$$

$$\cosh\rho \sinh\tau = uv_4, \tag{2.17}$$

$$\sinh\rho = \frac{1}{2u}\Big(1 - u^2(1 - v_4^2)\Big). \tag{2.18}$$

In the case of the EAdS, this two-dimensional part reduces to the two-dimensional section (u, v_4) of the Poincaré patch (u, v_1, v_2, v_3, v_4), whose metric is nothing but the EAdS_2, $u^{-2}du^2 + u^2dv_4^2$. as is easily seen by comparing this definition with the ordinary Poincaré coordinates of the EAdS_5 background.

Eqs. (2.15) show that in terms of the coordinates ρ and θ, the droplet plane D at $y = 0$ is described by two patches, either $\rho = 0$ or $\theta = 0$ as in the case of Lorentzian convention. Inside the region W, the former corresponds to $0 \leq x_1^2 - x_2^2 \leq r_0^2$, while the later to $x_1^2 - x_2^2 \geq r_0^2$. The EAdS ground state corresponds to the droplet where the first patch is completely filled. The droplets of general excited states have holes inside the first patch, and the second patch has nonzero occupied regions. If we take into account the Euclidean time coordinate τ, the boundary along which the two patches are sewn is the product space of hyperbola $x_1^2 - x_2^2 = r_0^2$ and a semi-circle defined by

$$\frac{1}{u^2} + v_4^2 = 1 \quad (u = \cosh\tau,\ v_4 = \tanh\tau). \tag{2.19}$$

The latter equation shows that the two-dimensional section in consideration reaches the conformal boundary $u \to \infty$ at $v_4 = \pm 1$. If we wish, this can be regarded again as a hyperbola by defining new variables $q = u,\ p = uv_4$ in terms of which the above equation is expressed as $q^2 - p^2 = 1$. We note that the semicircle coincides with the 'tunneling trajectory' [7, 8] which describes two-point functions of the BMN operators semi-classically using the GKPW relation. The time τ plays the role of affine parameter along the trajectory.

3. Scattering of droplets

3.1 Boundary S-matrix

Consider first a form of droplets which can be represented by the profile of their boundary in W

$$x_1 = a(\psi)\cosh\psi, \quad x_2 = a(\psi)\sinh\psi, \tag{3.1}$$

$$a(\psi) = r_0 + \tilde{a}(\psi). \tag{3.2}$$

The function $\tilde{a}(\psi)$ measures the deformation of the boundary profile of the ground-state droplet. In terms of the coordinate $\tilde{\psi} = \psi - \tau$ of the EAdS background, the above droplet should be reinterpreted as a τ-dependent form

$$x_1 = a(\tilde{\psi})\cosh(\tau + \tilde{\psi}), \quad x_2 = a(\tilde{\psi})\sinh(\tau + \tilde{\psi}). \tag{3.3}$$

In what follows, we rename $\tilde{\psi}$ by ψ for notational simplicity. The rationale for this interpretation of time dependent droplet is that the equation of small deformations around the EAdS background must be

$$\frac{d^2 x_i}{d\tau^2} = x_i, \tag{3.4}$$

which is obtained by our Wick rotation from the corresponding equation of AdS background as established in ref. [9] by studying the quantization of fluctuations around the AdS geometry. The ground state profile does not change under this motion, while the excited state profiles moves along the hyperbola, as is familiar in the treatment of collective motions of fermi liquid in the inverted harmonic oscillator potential in the semi-classical approximation of the $c = 1$ matrix model. The equation of supergravity around the EAdS background is obeyed by solutions obtained from these time dependent droplets at $y = 0$ through (2.11).

This implies that the scattering amplitude of the excited states of Euclideanized bubbling geometries from conformal boundary to conformal boundary in the bulk essentially coincides with that of the deformed droplets propagating from $\tau = -\infty$ to $\tau = +\infty$. This is also consistent with the fact that the energy of the droplet in supergravity is directly given [3] by the collective Hamiltonian of the droplet, together with the known observation that the symplectic structure of supergravity action also reduces [9] to that of the collective fields of the droplet.

Then, according to a general idea of the holographic correspondence, the GKPW relation, between bulk-boundary propagators and correlators on the boundary, we naturally expect that the S-matrix of droplets will be directly connected to the two-point functions of multi-trace 1/2-BPS operators of the (Euclideanized) SYM, under an appropriate mapping between the deformations of the ground-state profile and the set of those local operators.

There is a well-known method [10] for deriving the S-matrix of droplet in this picture which has been established in the context of $c = 1$ matrix model. To make the present paper reasonably self-contained, we briefly review the basic idea. Renaming x_i by $x_1 \to x$, $x_2 \to p$, the equation of motion for the profile in the phase space $(x, p_\pm(x, \tau))$ is

$$\frac{\partial}{\partial \tau} p_\pm = x - p_\pm \frac{\partial}{\partial x} p_\pm \tag{3.5}$$

where we interpret the momentum as a function of (x, τ) and the suffix $+$ and $-$ indicate two regions $p_+ > 0$ and $p_- < 0$, respectively: $p_\pm = \pm\sqrt{x^2 - a^2}$. The S-matrix is by definition given by considering the relation between two asymptotic regions $\tau \to \pm\infty$.

Following [10], we set for sufficiently large x

$$x = \mathrm{e}^q, \quad p_\pm = \pm \mathrm{e}^q \mp \mathrm{e}^{-q} \epsilon_\pm(q, \tau). \tag{3.6}$$

Since in these asymptotic regions we have $x \sim \frac{a(\psi)}{2} \mathrm{e}^{-\tau-\psi}$ and $\sim \frac{a(\psi)}{2} \mathrm{e}^{\tau+\psi}$ for $\tau \to \mp\infty$ respectively, the ϵ fields behave, with respect to the dependence on τ, as $\epsilon_\pm(q, \tau) \sim \epsilon_\pm(\tau \mp q)$. Hence the lapse of time between incoming (τ_i) and outgoing (τ_i) waves at a fixed large value of $x = \mathrm{e}^q$ satisfies

$$\tau_f - \tau_i = 2q + \log \frac{a(\psi)^2}{4}, \quad \epsilon_-(q, \tau_i) = \epsilon_+(q, \tau_f) = \frac{a(\psi)^2}{2} \tag{3.7}$$

which lead to a functional equation for asymptotic forms

$$\epsilon_+(\tau - q) = \epsilon_-\left(\tau - q - \log\frac{\epsilon_+(\tau - q)}{2}\right). \tag{3.8}$$

Or, using the fluctuating fields $\delta_\pm$ defined by $\frac{\epsilon_\pm(x)}{2} = c_0^2 + \delta_\pm(x)$, with c_0 being the scale of the static ground-state droplet,

$$\delta_+(x) = \delta_-(x - \log(c_0^2 + \delta_+(x)). \tag{3.9}$$

The solution is given as (see [11] for more details)

$$\delta_\pm(x) = -c_0^2\sum_{p=1}^{\infty}\frac{(-1)^p\Gamma(\pm\partial_x + p - 1)}{p!\Gamma(\pm\partial_x)}\left(\frac{\delta_\mp(x)}{c_0^2}\right)^p. \tag{3.10}$$

This can be interpreted as the relation between *in*- and *out*-fields of collective fields $\chi_\pm$, defined by

$$\delta_\pm \sim \sqrt{4\pi}(\partial_\tau \mp \partial_q)\chi_\pm, \quad \chi_\pm(\tau \mp q) = -\frac{i}{\sqrt{4\pi}}\int \alpha_\pm(\xi)\mathrm{e}^{i\xi(\tau\mp q)}\frac{d\xi}{\xi} \tag{3.11}$$

satisfying the canonical commutation relations,

$$[\alpha_\pm(\xi), \alpha_\pm(\xi')] = -\xi\delta(\xi' + \xi), \quad \alpha_\pm(-\omega)|0\rangle = 0 \quad \omega > 0. \tag{3.12}$$

In terms of the normal-mode operators in the momentum representation, we have

$$\alpha_\pm(\eta) = \sum_{p=1}^{\infty}\left(\frac{2}{c_0^2}\right)^{p-1}\frac{\Gamma(1\mp i\eta)}{\Gamma(2\mp i\eta - p)}\frac{1}{p!}\int d^p\xi\,\delta(\eta - \sum\xi_i)\Big(\prod\alpha_\mp(\xi_i)\Big) \tag{3.13}$$

and the S-matrix in the classical (tree) approximation is given by

$$S(\sum\omega_i \to \sum\omega_i') = \langle 0|\prod_i \alpha_-(-\omega_i')\prod_j \alpha_+(\omega_j)|0\rangle. \tag{3.14}$$

For example, for $n \to 1$ scattering ($n \geq 2$), the S-matrix elements are, up to the delta-function of energy conservation ($\omega = \omega_1 + \cdots \omega_n$) which we will always suppress in what follows,

$$\langle 0|\alpha_-(-\omega)\alpha_+(\omega_1)\cdots\alpha_+(\omega_n)|0\rangle \Rightarrow \left(\frac{2}{c_0^2}\right)^{n-1}(-i\omega)(-i\omega - 1)\cdots(-i\omega - n + 2)\,\omega_1\cdots\omega_n. \tag{3.15}$$

In applying these results to our case, we have to Wick-rotate the momentum (=±energy on the mass-shell) as $\xi = \omega \to iJ$ with J being the R-charge, in order to take into account our prescription of Euclideanized bubbling geometries. Note that this procedure effectively makes the *coupling constant*, $1/c_0^2$, pure imaginary, apart from an overall factor i which can be absorbed into Wick rotation of the δ-function of energy conservation.

3.2 Hamiltonian formalism

We finally recall that the same S-matrix elements as above are obtained by the Hamiltonian formalism of collective field theory [12]. The equation (3.5) can be recast in the Hamiltonian form

$$\partial_\tau p_\pm = i[H, p_\pm] \tag{3.16}$$

by defining the effective Hamiltonian and the commutation relations as

$$H = \int dx \left[\left(\frac{p_+^3}{6} - \frac{(x^2+\mu)p_+}{2} \right) - \left(\frac{p_-^3}{6} - \frac{(x^2+\mu)p_-}{2} \right) \right], \tag{3.17}$$

$$[p_\pm(x,\tau), p_\pm(x',\tau)] = \mp i\delta'(x-x'). \tag{3.18}$$

Here, μ is an arbitrary constant, corresponding to an integration constant for the equation of motion. Note also that we use the usual Lorentzian convention in writing down the equations of motion, by interpreting (3.4) as being due to the inverted harmonic potential $V(x) = -x^2/2$. Actually, there is an ambiguity whether we take positive μ or negative one. Here we choose the positive convention for definiteness. This choice has an advantage in that the interaction Hamiltonian is not singular. The negative μ, which looks more natural in view of the form of the profile function, would give a singular interaction Hamiltonian. Remarkably, however, there exists a duality that *both* give the *same* S-matrix with suitable regularization for the negative choice.

To make the system look more like a usual canonical system, we define the shifted fields $\tilde{\phi}_\pm$ (the sign of μ becomes relevant here):

$$p_\pm(x,\tau) = \pm(\sqrt{x^2+\mu} + \tilde{\phi}_\pm(x,\tau)), \tag{3.19}$$

$$H = \int dx \left[\frac{1}{2}\sqrt{x^2+\mu}(\tilde{\phi}_+^2 + \tilde{\phi}_-^2) + \frac{1}{6}(\tilde{\phi}_+^3 + \tilde{\phi}_-^3) \right], \tag{3.20}$$

which reduces, by further making a change of variables $x = \mu \sinh\sigma$, $\tilde{\phi}_\pm = \left|\frac{d\sigma}{dx}\right|\phi_\pm$, to

$$H = \int_0^\infty d\sigma \left[\frac{1}{2}(\phi_+^2 + \phi_-^2) + \frac{1}{6}\left|\frac{d\sigma}{dx}\right|^2 (\phi_+^3 + \phi_-^3) \right]. \tag{3.21}$$

Here we have subtracted a (field-independent) c-number contribution. Then, using the normal-mode expansion in the interaction representation,

$$\phi_\pm(\sigma,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty d\xi \mathrm{e}^{-i\xi(\tau\mp\sigma)}\alpha(\xi) \tag{3.22}$$

with $[\alpha(\xi), \alpha(\xi')] = -\omega\delta(\xi+\xi')$ which can be identified with the normal-mode operators of $\chi_\pm$ introduced above in the asymptotic region $|\sigma| \sim q \to \infty$, we obtain

$$H = H_2 + H_3(\tau), \tag{3.23}$$

$$H_2 = \int_0^\infty d\xi\, \alpha(\xi)\alpha(-\xi), \tag{3.24}$$

$$H_3(\tau) = \frac{1}{6}\int_{-\infty}^\infty d^3\xi\, f(\xi_1+\xi_2+\xi_3)\, \mathrm{e}^{-i(\xi_1+\xi_2+\xi_3)\tau}\alpha(\xi_1)\alpha(\xi_2)\alpha(\xi_3) \tag{3.25}$$

with

$$f(\xi) = \int_{-\infty}^{\infty} d\sigma \, \frac{1}{\mu \cosh^2 \sigma} \mathrm{e}^{i\xi\sigma} = \frac{2\pi\xi}{\mu \sinh \pi\xi}. \tag{3.26}$$

Note that the range of the σ-integral becomes the whole real axis by combining $\phi_\pm$ contributions into a single integral, even though the original range was the half real axis $0 < \sigma < \infty$. If we choose negative μ, the form factor in H_3 would have been $1/\sinh^2 \sigma$ instead of $1/\cosh^2 \sigma$. Using this interaction Hamiltonian, it is straightforward to compute the S-matrix in perturbative expansion in $1/\mu \propto 2/c_0^2$. The agreement of the results with the polynomial form exhibited in (3.13) and (3.15) has been confirmed in [13][14][15], together with some extensions to quantum corrections. This is somewhat miraculous in view of the presence of a nontrivial form factor $f(\xi)$ in the interaction Hamiltonian (3.25).

4. Comparison with the two-point functions of multi-trace operators

Let us recall the structure of a representative set of the chiral primary 1/2-BPS operators on the Yang-Mills side, which are characterized SO(4) symmetry and the conformal dimensions $\Delta = J$,

$$\mathcal{O}^J_{(J_1,J_2,\ldots,J_n)}(x) \equiv \mathrm{Tr}\Big(Z(x)^{J_1}\Big)\mathrm{Tr}\Big(Z(x)^{J_2}\Big)\cdots\mathrm{Tr}\Big(Z(x)^{J_n}\Big), \quad J = \sum_i J_i \tag{4.1}$$

where $Z = (\phi_5 + i\phi_6)/\sqrt{2}$ is the complex scalar field with a unit R-charge $J = 1$ with respect to the rotation in the 5-6 plane. Due to the non-renormalization property, two-point correlation functions of these operators with their conjugate set of operators constructed in terms of $\overline{Z} = (\phi_5 - i\phi_6)/\sqrt{2}$ are given by the free-field results,

$$\langle \overline{\mathcal{O}}^J_{(J'_1,J'_2,\ldots,J'_m)}(x')\,\mathcal{O}^J_{(J_1,J_2,\ldots,J_n)}(x)\rangle = F(\{(J'),(J)\},N)D_4(x,x')^J \tag{4.2}$$

where $D_4(x,x') \propto |x-x'|^{-2J}$ with $J = \sum_i J_i = \sum_i J'_i$ is the massless free-field propagator in 4 dimensions and the function $F(\{(J'),(J)\},N)$ is determined by the free-field contraction among the indices of scalar fields ϕ_i between $O^J(x)$ and $\overline{O}^J(x')$. Obviously, the function F is completely independent of spacetime coordinates. Furthermore, it is independent of the spacetime dimensions and signature. The double Wick-rotation we have discussed in the previous section requires us to Wick-rotate the angle coordinate in the 5-6 plane. In terms of the above scalar fields, it amounts to rotating ϕ_6 to pure imaginary axis, $i\phi_6$, and hence replace Z ($\overline{Z}$) by $Z = (\phi_5 - \phi_6)/\sqrt{2}$ ($\overline{Z} = (\phi_5 + \phi_6)/\sqrt{2}$) with ϕ_6 now being quantized with *negative* metric. The crucial property $\langle Z\,Z\rangle = \langle \overline{Z}\,\overline{Z}\rangle = 0$ is preserved under this procedure, and hence this rotation does not affect the above final form of the correlators.

For the specific purpose of the present section, comparison of the S-matrix elements with the correlators, it is sufficient to study only the function $F(\{(J'),(J)\},N)$ which is independent of possible choices of matrix models. Let us therefore suppress the spacetime coordinates and, hence, spacetime dependent factor such as $D_4(x,x')$ in what follows. The simplest case $m = n = 1$, we have

$$\langle \mathrm{Tr}\overline{Z}^J \mathrm{Tr}Z^J\rangle = JN^J. \tag{4.3}$$

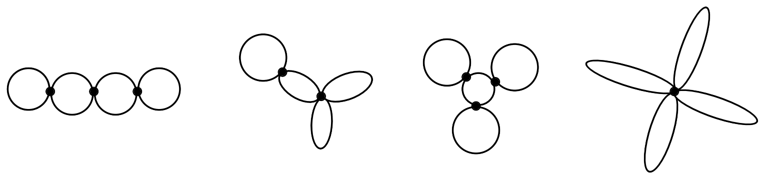

Figure 3: Four types of planar contractions of multiple traces into a single trace for $1 \leftrightarrow 4$ scattering.

The results of the previous section suggests that this trivial two-point function should be interpreted as the trivial $(1 \to 1)$ S-matrix element

$$\langle 0|\alpha_-(-iJ)\alpha_+(iJ)|0\rangle = J. \tag{4.4}$$

As before we have suppressed an overall δ-function factor $-i\delta(J-J) = -i\delta(0)$ of momentum conservation, which is common for all S-matrix elements. Then, the natural normalization for making correspondence between the S-matrix elements of the droplet and the correlators is $\alpha(iJ) \leftrightarrow \frac{1}{N^{J/2}}\mathrm{Tr}Z^J$ for incoming states and $\alpha(-iJ) \leftrightarrow \frac{1}{N^{J/2}}\mathrm{Tr}\overline{Z}^J$ for outgoing states.

Once single-trace operators are related to single-particle states, the multi-trace operators must be interpreted as multi-particle states. Under this interpretation, let us study some examples of $(n \to 1)$ correlators in the leading planar approximation in the large N limit. We present the results of graphical computations for $n = 2, 3, 4$ ($J = \sum_{i=1}^{n} J_i$).

$$\begin{aligned} F(J,\{J_1,J_2\},N)_{\text{planar}} &= JJ_1J_2N^{-1}, \\ F(J,\{J_1,J_2,J_3\},N)_{\text{planar}} &= J\Big(J_1J_2J_3(J_1-1)+J_1J_2J_3(J_2-1)+J_1J_2J_3(J_3-1)+2J_1J_2J_3\Big)N^{-2} \\ &= J_1J_2J_3J(J-1)N^{-2} \end{aligned} \tag{4.5}$$

where for $n = 3$ the first 3 contributions come from 'chain'-type diagrams in which the 3 traces of the initial state connected like a chain of 3 beads, while the last one comes from a 'clover'-type diagram in which all of the 3 traces are connected simultaneously at a single point.

$$\begin{aligned} F(J,\{J_1,J_2,J_3,J_4\},N)_{\text{planar}} =J\Big[&2\Big(J_1J_2(J_2-1)J_3(J_3-1)J_4 \\ &+J_1J_3(J_3-1)J_4(J_4-1)J_2+J_1J_2(J_2-1)J_4(J_4-1)J_3 \\ &+J_2J_1(J_1-1)J_4(J_4-1)J_3+J_3J_1(J_1-1)J_2(J_2-1)J_4 \\ &+J_2J_1(J_1-1)J_3(J_3-1)J_4\Big) \\ &+6\Big(J_1J_2J_3(J_3-1)J_4+J_2J_3J_1(J_1-1)J_4+J_1J_2(J_2-1)J_3J_4 \\ &+J_1J_2J_3J_4(J_4-1)\Big) \\ &+\Big(J_1(J_1-1)(J_1-2)J_2J_3J_4+J_1J_2(J_2-1)(J_2-2)J_3J_4 \\ &+J_1J_2J_3(J_3-1)(J_3-2)J_4+J_1J_2J_3J_4(J_4-1)(J_4-2)\Big) \\ &+6J_1J_2J_3J_4\Big]N^{-3} = J_1J_2J_3J_4J(J-1)(J-2)N^{-3} \end{aligned} \tag{4.6}$$

where, similarly with the case $n = 3$, the first round bracket comes from chain diagrams of 4 beads, the second from three-leaf clover with one of the leafs being a chain of two beads, the third from 'sunflower'-type diagrams in which three traces are connected to one central trace at three separate points, and the last from 4-leaf clover diagrams in which all of 4 traces are connected at one point, respectively. See Fig. 3 for illustration. These results precisely match to the corresponding S-matrix elements (3.15), provided the expansion parameter is related by $i2/c_0^2 \to 1/N$.

The same results as above can also be obtained using the exact general formulas [17], which are expressed as linear combinations of the ratios of Gamma functions, such as (say $n = 1, 2$)

$$\langle \mathrm{Tr}(\overline{Z}^J)\mathrm{Tr}(Z^J)\rangle = \frac{1}{J+1}\Big(\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)}\Big), \tag{4.7}$$

$$\begin{aligned}\langle \mathrm{Tr}(\overline{Z}^J)\mathrm{Tr}(Z^{J_1})\mathrm{Tr}(Z^{J_2})\rangle &= \frac{1}{J+1}\Big(\frac{\Gamma(N+J_1+J_2+1)}{\Gamma(N)} + \frac{\Gamma(N+1)}{\Gamma(N-J_1-J_2)} \\ &\quad -\frac{\Gamma(N+J_1+1)}{\Gamma(N-J_2)} - \frac{\Gamma(N+J_2+1)}{\Gamma(N-J_1)}\Big), \quad \text{etc}\end{aligned} \tag{4.8}$$

for these type of correlators. See the Appendix. These exact formulas will also be used for a study of the higher-genus effect in the next section.

The computation of higher-point cases and more general configurations of initial and final states becomes increasingly cumbersome. To convince ourselves the validity of agreement further, it is useful to consider the limit of large momentum J. Let us define a unitary scattering operator $S = \exp\mathcal{V}$ by

$$S\alpha_+(\eta)S^{-1} = \sum_{n=0} \frac{1}{n!} \overbrace{[\mathcal{V}, [\mathcal{V}, [\cdots, [\mathcal{V}}^{n\ times}, \alpha_+(\eta) \overbrace{]\cdots]]]}^{n\ times} = \alpha_-(\eta). \tag{4.9}$$

It should be kept in mind that here the commutators should be understood as Poisson brackets, since we are in the tree approximation where the operator ordering is irrelevant. In the limit of large momentum, we can replace the (3.13) by ($\kappa = 2/c_0^2$)

$$\alpha_\pm(\eta) = \sum_{p=1}^{\infty} \kappa^{p-1} \frac{(\mp i\eta)^{p-1}}{p!} \int d^p\xi \delta(\eta - \sum \xi_i) \Big(\prod \alpha_\mp(\xi_i)\Big). \tag{4.10}$$

Then, we find that the scattering operator can be written in a simple closed form as

$$S = \exp\Big[i\frac{\kappa}{6}\int d\xi_1 \int d\xi_2 \int d\xi_3\, \delta(\xi_1+\xi_2+\xi_3)\alpha_+(\xi_1)\alpha_+(\xi_2)\alpha_+(\xi_3)\Big] \tag{4.11}$$

For example, the second term $n = 1$ $(p = n + 1)$ is

$$[i\frac{\kappa}{6}\int d\xi_1 \int d\xi_2\, \alpha_+(\xi_1)\alpha_+(\xi_2)\alpha_+(-\xi_1-\xi_2), \alpha_+(\eta)] = \frac{i\eta\kappa}{2}\int d\xi_1\, \alpha_+(\xi_1)\alpha_+(\eta-\xi_1) \equiv O_1$$

The third term (n=2) is equal to

$$\frac{1}{2}\left[i\frac{\kappa}{6}\int d\xi_1\int d\xi_2\,\alpha_+(\xi_1)\alpha_+(\xi_2)\alpha_+(-\xi_1-\xi_2),\frac{i\eta\kappa}{2}\int d\xi_1\,\alpha_+(\xi_1)\alpha_+(\eta-\xi_1)\right]$$
$$=\frac{i\kappa^2\eta}{2}\int d\xi_1\int d\xi_2\,(i\xi_1)\alpha_+(\xi_1)\alpha_+(\xi_2)\alpha_+(\eta-\xi_1-\xi_2)$$
$$=\frac{i\eta\kappa^2}{2}\int d\xi_1\int d\xi_2\int d\xi_3\,\delta(\xi_1+\xi_2+\xi_3-\eta)\,\frac{1}{3}\,i(\xi_1+\xi_2+\xi_3)\alpha_+(\xi_1)\alpha_+(\xi_2)\alpha_+(\xi_3)$$
$$=\frac{(i\eta)^2\kappa^2}{6}\int d\xi_1\int d\xi_2\int d\xi_3\delta(\xi_1+\xi_2+\xi_3-\eta)\,\alpha_+(\xi_1)\alpha_+(\xi_2)\alpha_+(\xi_3)\equiv O_2 \tag{4.12}$$

To establish the general terms in the claimed form, we can proceed by mathematical induction. Suppose the n-th term in the expansion is given by

$$O_n\equiv\frac{(i\eta)^n\kappa^n}{(n+1)!}\int\cdots\int d\xi_1\cdots d\xi_{n+1}\,\delta(\xi_1+\cdots+\xi_{n+1}-\eta)\alpha_+(\xi_1)\cdots\alpha_+(\xi_{n+1}).$$

Then by similar manipulation as for $n=2$, we can check that the next term $n+1$ is indeed given by

$$\frac{i}{(n+1)}\left[i\frac{\kappa}{6}\int d\xi_1\int d\xi_2\,\alpha_+(\xi_1)\alpha_+(\xi_2)\alpha_+(-\xi_1-\xi_2),O_n\right]=O_{n+1}.$$

When we use the Euclidean convention from the beginning, we obtain the same result by making a replacement $\mathcal{V}\to\mathcal{V}_3$.

$$\mathcal{V}_3=\frac{1}{2N}\sum(\alpha(J_1+J_2)\alpha(-J_1)\alpha(-J_2)+\alpha(J_1)\alpha(J_2)\alpha(-J_1-J_2)) \tag{4.13}$$

by restricting the value of (purely imaginary) momentum by discrete integers J_i's. The vertex is essentially the time-integrated interaction Hamiltonian (3.25), restricted on the mass-shell, of collective field theory

$$\mathcal{V}_3\sim\int\frac{d\tau}{2\pi}\,H_3(\tau).$$

In fact, it is not surprising that in the limit of large momentum, the intermediate states behave almost like on-mass-shell states.

Interestingly, precisely the same formula (4.13) has previously been proposed in [6] which seems to be motivated purely on a combinatorial basis that this overlap-type 3-point vertex is simulating the structure of splitting and joining of matrix traces.[3] The suggestion made in the last reference was that this formula should be valid for arbitrary genus in the BMN limit ($J,N\to\infty, g_2=J^2/N$ =fixed). Our results can thus be regarded as a generalization of this proposal for arbitrary finite J at least in the planar approximation, by making clear the basis for the correspondence between the $c=1$ matrix model and the bubbling geometry. In particular our S-matrix interpretation implies that for finite J the above vertex would not be appropriate for evaluation of higher then 3-point correlators. For that one should uses the nontrivial c=1 string vertex as in [13].

[3]One of the present author (T.Y.) thanks K. Okuyama for a discussion on his work.

5. Droplet scattering from the viewpoint of the $c = 1$ matrix model

5.1 $c = 1$ scattering interpretation of the correlators

Since the function f does not depend on spacetime dimensions, we can consider a matrix quantum mechanics to represent the same correlators. Traditionary, we use a complex matrix for this purpose with special constraint called the 'lowest Landau-level condition', as originally discussed in [1] and followed by most literature related to this subject.

From the reduction, it is clear that one can start with the ordinary Hermitian matrix model (of Lorentzian signature), $\int d\tau \, \frac{1}{2}\mathrm{Tr}\left[\left(\frac{dM}{d\tau}\right)^2 - M^2\right]$, by considering the correlators in the following normal-order prescription,

$$\langle :\mathcal{O}^{J}_{(J'_1,\dots,J'_m)}(\tau')_M : :\mathcal{O}^{J}_{(J_1,\dots,J_n)}(\tau)_M :\rangle = F(\{(J'),(J)\},N) D_1(\tau,\tau')^J \tag{5.1}$$

where $D_1(\tau,\tau') \propto \exp(i|\tau-\tau'|)$ and

$$\mathcal{O}^{J}_{(J_1,\dots,J_n)}(\tau_1)_M \equiv \mathrm{Tr}\Big(M^{J_1}\Big)\mathrm{Tr}\Big(M^{J_2}\Big)\cdots\mathrm{Tr}\Big(M^{J_n}\Big). \tag{5.2}$$

The normal product symbol $:\cdots:$ indicates that no contraction is allowed inside. This interpretation was emphasized in ref. [5] to be useful for extending discussions to more general 1/2-BPS operators in which the other SO(6) states than (4.1) appear on an equal footing.

In our case, it is more natural to consider the model with inverted harmonic potential of negative sign, in order to utilize the standard $c = 1$ matrix model to which the Euclideanized bubbling geometries fit well,

$$S_{c=1} = \int d\tau \, \frac{1}{2}\mathrm{Tr}\left[\left(\frac{dM}{d\tau}\right)^2 + M^2\right] \tag{5.3}$$

and consider the operators

$$\Pi_\pm = (M \pm \dot{M})/\sqrt{2}. \tag{5.4}$$

The correlators are then given as

$$\langle \mathcal{O}^{J}_{(+;J'_1,\dots,J'_m)}(\tau')\, \mathcal{O}^{J}_{(-;J_1,\dots,J_n)}(\tau)\rangle = F(\{(J'),(J)\},N) D_1(\tau,\tau')^J \tag{5.5}$$

with

$$\mathcal{O}^{J}_{(\pm;J_1,\dots,J_n)}(\tau_1) \equiv \mathrm{Tr}\Big(\Pi_\pm^{J_1}\Big)\mathrm{Tr}\Big(\Pi_\pm^{J_2}\Big)\cdots\mathrm{Tr}\Big(\Pi_\pm^{J_n}\Big). \tag{5.6}$$

Note that we now have $D_1(\tau,\tau') \propto \exp(-|\tau-\tau'|)$. The normal-order prescription is automatically taken into account since $\langle \Pi_+\Pi_+\rangle = \langle \Pi_-\Pi_-\rangle = 0$. In the language of the complex matrix model of ref. [1], the matrix operators $\Pi_\pm$ are related to the following canonical decomposition

$$Z = \frac{1}{\sqrt{2}}(A^\dagger + B), \quad \bar{Z} = \frac{1}{\sqrt{2}}(A + B^\dagger)$$

with $A, A^\dagger$ being replaced by $\Pi_\pm$. The lowest Landau level condition amounts to eliminating the additional canonical pair $(B, B^\dagger)$.

For evaluating the correlators, we can use the collective-field representation. Remember that in the planar approximation of matrix models, the collective field theory is essentially a phase-space representation of free-fermion liquid. For our purpose, it is most convenient to use the coherent-state representation of the phase space in which $\Pi_-(\Pi_+) \sim A^\dagger(A)$ are regarded as generalized coordinate (z) and momentum (α), respectively.

$$\begin{aligned} \mathrm{Tr}\left(\Pi_-^J\right) &\to \int \frac{dz}{2\pi} \int^\alpha d\alpha\, z^J = \int \frac{dz}{2\pi}\, z^J \alpha(z) = \alpha_{-J}, \\ \mathrm{Tr}\left(\Pi_+^J\right) &\to \int \frac{dz}{2\pi} \int^\alpha d\alpha\, \alpha^J = \int \frac{dz}{2\pi}\, \frac{\alpha(z)^{J+1}}{J+1}. \end{aligned} \tag{5.7}$$

It is relevant to note that there is also a (dual) coherent state representation in which

$$\begin{aligned} \mathrm{Tr}\Pi_-^J = \beta_J &= \int \frac{dz}{2\pi} z^{-J} \beta(z), \\ \mathrm{Tr}\left(\Pi_+^J\right) &= \int \frac{dz}{2\pi} \frac{\beta(z)^{J+1}}{J+1}. \end{aligned} \tag{5.8}$$

In the (analytically continued) scattering picture, the operators α_J or β_J will be seen to coincide with *in* (*out*) fields respectively. The existence of the dual representation can be related to the freedom of two seemingly different choices for the sign of μ, appearing in the construction of the collective Hamiltonian as has already been alluded to in section 3.

The correlator then becomes

$$\langle 0| \int \frac{dz_1}{2\pi} \frac{\alpha(z_1)^{J_1'+1}}{J_1'+1} \cdots \int \frac{dz_m \alpha(z_m)^{J_m'+1}}{J_m'+1}\; \alpha_{-J_1}\alpha_{-J_2}\cdots\alpha_{-J_n}|0\rangle.$$

In this picture, the operators $\mathrm{Tr}\left(\Pi_-^J\right)$ are simply creation operators while the operators $\mathrm{Tr}\left(\Pi_+^{J'}\right)$ are nontrivial polynomials. It is a theorem that we shall prove below that they are generated by the analog of the $(c=1)$ S-matrix operator. We have

$$\mathrm{Tr}\left(\Pi_+^J\right) = S^{-1}\alpha_J S.$$

Consequently, the above matrix elements are

$$\langle \quad \rangle = \langle 0|\alpha_{J_1'}\cdots\alpha_{J_m'} S \alpha_{-J_1}\alpha_{-J_2}\cdots\alpha_{-J_n}|0\rangle$$

To demonstrate that S is nothing but the S-matrix of the $c=1$ theory, we recall the analysis through collective field theory [12]. One considers

$$\mathrm{Tr}\Pi_+^l \to \mathrm{Tr}\left((P+X)/\sqrt{2}\right)^{ik} = W_{ik}^{(+)}, \tag{5.9}$$

which is represented as

$$W_{ik}^{(+)} = \int \frac{dx}{2\pi} \left(\frac{\alpha_+^{ik+1}}{ik+1} - \frac{\alpha_-^{ik+1}}{ik+1} \right). \tag{5.10}$$

This operator has an exact time evolution, being an eigenstate of the Hamiltonian $H = \mathrm{Tr}(P^2 - X^2)/2 = \mathrm{Tr}(\Pi_-\Pi_+ + \Pi_+\Pi_-)/2$. A multiplication by e^{-ikt} makes it into a constant of motion. Evaluating this operator at early $(t \to -\infty)$ and late $(t \to +\infty)$ times gives the S-matrix of the theory. One has

$$\alpha_\pm(x) = \pm x \mp \frac{1}{x}\left(\mu \mp \hat{\alpha}_\pm(\tau)\right)$$

with $x \sim \sqrt{\mu/2}e^\tau$. At early time, we will have only a left moving wave packet given by $\hat{\alpha}_-(t+\tau)$ and at late time we have a right moving packet defined by $\hat{\alpha}_+(t-\tau)$. The evaluation of W_{ik} was performed in detail in [16]. At early time, with the left-moving packet,

$$W_{ik} \to -\frac{(\sqrt{2\mu})^{ik+1}}{ik+1}\int \frac{d\tau}{8\pi}\, e^{-ik\tau}\sum_{p=1}^{\infty}\frac{(ik+1)!}{(ik+1-p)!p!}\left(\frac{\alpha_-}{\mu}\right)^p .$$

At late time (with a right moving packet) only a single mode survives in the limit giving

$$W_{ik} \to \left(\sqrt{2\mu}\right)^{ik+1}\int \frac{d\tau}{8\pi}\, e^{ik\tau}\,\frac{\alpha_+(\tau)}{\mu}.$$

Consequently, one obtains a relationship between the outgoing modes and the incoming ones

$$\int d\tau e^{ik\tau}\hat{\alpha}_+(\tau) = -\int \frac{d\tau}{ik+1}\, e^{-ik\tau}\left(1+\frac{\hat{\alpha}_-}{\mu}\right)^{1+ik} \tag{5.11}$$

which is equivalent with (3.13) obtained on the basis of a droplet picture on the bulk side. The $c=1$ S-matrix is defined by the transformation $\hat{\alpha}_+ = S^{-1}\hat{\alpha}_- S$ The identification

$$\hat{\alpha}_-(z) \leftrightarrow \alpha(z), \hat{\alpha}_+(z) \leftrightarrow \beta(z)$$

then establishes the statement that the 1/2-BPS correlators coincide with the scattering amplitudes of the c=1 theory.

We emphasize that this holds without the so-called leg factors which appear in the physical interpretation of the c=1 matrix model . Also the analysis given above was done in the tree (or planar) approximation.

We note that this correspondence is related to the observation of [18] where an identification between finite temperature c=1 amplitudes and the normal matrix integral was described. In contrast to the AdS/CFT interpretation that we were concerned with in the present work, the correspondence discussed in [18] (see also [19]) gives a special integration measure on the complex matrix model side such that equivalence is achieved with finite temperature correlation functions.

5.2 Case of higher-genus

Thus far we have given a basis for the correspondence between the 1/2-BPS correlators and the $c = 1$ S-matrix from both sides of bulk and boundary theories at the planar approximation. It is then of interest to see to what extent the correspondence will be valid beyond this approximation. On the bulk side, evaluating string-loop effects in the (E)AdS background is an unsolved problem. On the side of the gauge theory and the matrix models, the usual fermion representation based on the positive harmonic potential gives a rigorous definition of the correlators for arbitrary finite N and integer J. In principle, there must be a version of $c=1$ fermion representation with negative harmonic potential, which gives the identical function F for finite N and J. It would require a special regularization in dealing with the negative-sign harmonic potential by appropriately taking into account the differences discussed in section 2. We will not elaborate on such a direction in the present note, since the problem is rather a matter of interpretation.

Instead, we expect on the ground of universality that the S-matrix of the $c=1$ model defined by the usual double scaling limit gives the right answer for large N and J. In the tree approximation, the large J limit was not necessary, as is reasonable since the momentum along the R-charge direction must be conserved and internal momenta in the tree approximation are fixed by external momenta. It is natural to expect that the discreteness of J would be washed out in the limit of large J even for internal momenta.

In the rest of this section, we present a piece of evidence for this expectation by studying the simplest nontrivial case, $1 \to 1$ amplitude, of higher-genus effect in the leading large-J approximation. In the coherent state representation, one has the Hamiltonian for a single fermion

$$h = \left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) - \mu. \tag{5.12}$$

The wave-function can be taken as functions of a_- or (the dual picture) of a_+. One has the simple transform from one basis to another

$$\psi_-(a_+) = \frac{1}{\sqrt{2\pi}} \int da_- e^{-a_+a_-}\psi_+(a_-). \tag{5.13}$$

The wave-functions both obey

$$\frac{1}{2}\left(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+\right)\psi_\pm = \left(i\partial_\tau + \mu\right)\psi_\pm. \tag{5.14}$$

In the coherent state representation with $\hat{a}_- = a\,\hat{a}_+ = \partial/\partial a$, one has the equation

$$\left(a\frac{\partial}{\partial a} + \frac{1}{2} - \mu\right)\psi_{+,k}(a) = k\psi_{+,k}(a). \tag{5.15}$$

From the viewpoint of the fermion phase-space (x,p), creation-annihilation coordinates are the null plane coordinates $a_\pm \to x_\pm \equiv (p \pm x)/\sqrt{2}$. The discussion of scattering theory in the fermionic picture can be found in [20, 21].

The fermionic wave-functions $\psi_\pm(x_+)$ obey the analogous Schrödinger eqs. The equation is solved by

$$\psi_{+,k}(x_-) = A_k x_-^{-i(k-\mu)}\Theta(x_-) + B_k(-x_-)^{-i(k-\mu)}\Theta(-x_-) \tag{5.16}$$

with an analogous solution for ψ_-. The (Gaussian) transform then leads to the relation between the *in* and *out* fermion wave-function with the following reflection coefficient (see [22])

$$R(k-\mu) = \frac{\Gamma\left(\frac{1}{2} - i(k-\mu)\right)}{\sqrt{2\pi}}\left[e^{i\frac{\pi}{2}\left(\frac{1}{2}-(k-\mu)\right)} + \gamma\, e^{-i\frac{\pi}{2}(k-\mu)}\right] \tag{5.17}$$

where γ specifies the boundary conditions.

In conformal field notation, the fermions have the expansion

$$\psi(z) = \sum z^{-n}\psi_n, \tag{5.18}$$

$$\psi^+(z) = \sum z^{-n}\psi_n^+ \tag{5.19}$$

with $n\epsilon Z + \frac{1}{2}$, and

$$\{\psi_n, \psi_m^+\} = \delta_{n+m,0}. \tag{5.20}$$

The reflection coefficient is defined by

$$\psi_{-n}^{\rm out} = (R\psi)_{-n}^{\rm in} \tag{5.21}$$

where $\psi^{\rm in}(z) \equiv \psi_-(z), \psi^{\rm out}(z) \equiv \psi_+(z)$. Standard Bosonization formulas then define the corresponding in (0ut) bosonic fields. To make contact with the previous notation we have

$$\psi_\pm^\dagger(z)\psi_\pm(z) = \alpha_\pm(z) \tag{5.22}$$

The $\hat{S}$-matrix is then given by

$$\langle 0| \prod_{k=1}^m \sum_n \left(R^*\psi^\dagger\right)_n (R\psi)_{l_k+n}\ \alpha_{-j_1}\alpha_{-j_2}\cdots\alpha_{-j_n}|0\rangle. \tag{5.23}$$

To compare this result with the 1/2-BPS correlators, let us use the known results of genus expansion of two-point amplitude given in [20]. Using their notation, the contributions up to 3 loops in the limit of large momentum q are given as

$$R(q,-q)_1 \approx -\frac{1}{24}q^5, \quad R(q,-q)_2 \approx \frac{3}{5760}q^9, \quad R(q,-q)_3 \approx -\frac{9}{2903040}q^{13} \tag{5.24}$$

where we have omitted a factor $(\Gamma(-|q|))^2\ \mu^{|q|}$ multiplying each term for brevity.

On the other hand, the exact 2-point correlator reads (see (4.7))

$$G(J) = \frac{1}{J+1}\left[\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)}\right].$$

Using the expansion formula for the ratio of Gamma functions as discussed in the Appendix, we find

$$G(J) \approx \frac{N^{J+1}}{J+1}\sum_{n=0}^\infty \frac{1}{N^n}\binom{\beta}{n}\left[\left(\frac{J}{2}\right)^n - \left(-\frac{J}{2}\right)^n\right].$$

Only the odd $n = 2k+1$ terms are nonzero, giving the expansion

$$N^{-J}G(J) \approx \sum_{k=0}^\infty \frac{2}{JN^{2k}}\binom{J}{2k+1}\left(\frac{J}{2}\right)^{2k+1} \approx \frac{2N}{J}\sinh\frac{J^2}{N},$$

or

$$N^{-J}G(J) \approx J\left[1 + \frac{1}{N^2}\frac{J^4}{3!2^2} + \frac{1}{N^4}\frac{J^8}{5!2^4} + \frac{1}{N^6}\frac{J^{12}}{7!2^6} + \cdots\right]. \tag{5.25}$$

The odd denominators give coefficients 24, 1920, 322560 agreeing with the three coefficients in the $c = 1$ theory, up to an overall factor i which should be combined to the δ-function of energy conservation as in the planar cases and also to sign factor $(-1)^n$ for n loops.

We can confirm that this agreement is valid to all orders. For this purpose, it is useful to recall the exact expression given in [20] for the matrix model two-point function.

$$\frac{\partial}{\partial\mu}R(q,-q) = \Gamma(-|q|)^2 \ \mathrm{Im}\left\{ e^{i\pi|q|/2}\left(\frac{\Gamma\left(|q|+\frac{1}{2}-i\mu\right)}{\Gamma\left(\frac{1}{2}-i\mu\right)} - \frac{\Gamma\left(\frac{1}{2}-i\mu\right)}{\Gamma\left(-|q|+\frac{1}{2}-i\mu\right)}\right)\right\} \tag{5.26}$$

which gives the expansion

$$R(q,-q) = (q\Gamma(-|q|))^2 \ \mu^{|q|}\left\{\frac{1}{|q|} - \mu^{-2}\cdot\frac{1}{24}\cdot(|q|-1)(q^2-|q|-1)+\cdots\right\} \tag{5.27}$$

In the large q limit, one notices the identity

$$\frac{\partial}{\partial\mu}R(q,-q) = \mu^{-1}|q|\,R(q,-q)$$

which follows from the fact that the factor $\mu^{|q|}$ gives the dominant effect, remembering that we keep only the leading large q term in each order of $1/\mu^2$ expansion. Consequently at large q (only) we can use the formula

$$R(q,-q) \approx \frac{1}{|q|}\mu\frac{\partial}{\partial\mu}R(q,-q), \tag{5.28}$$

and we have

$$\begin{aligned}\Gamma(-|q|)^{-2}R(q,-q) \approx \frac{1}{2|q|i}&\left\{e^{i\frac{\pi}{2}|q|}\left(\frac{\Gamma(q+\frac{1}{2}-i\mu)}{\Gamma(\frac{1}{2}-i\mu)} - \frac{\Gamma(\frac{1}{2}-i\mu)}{\Gamma(-q+\frac{1}{2}-i\mu)}\right)\right.\\ &\left.-e^{-i\frac{\pi}{2}|q|}\left(\frac{\Gamma(q+\frac{1}{2}+i\mu)}{\Gamma(\frac{1}{2}+i\mu)} - \frac{\Gamma(\frac{1}{2}+i\mu)}{\Gamma(-q+\frac{1}{2}+i\mu)}\right)\right\}.\end{aligned}$$

Using the same expansion formula for the two terms inside the round bracket as before, one generates at large q the terms

$$\pm\frac{(\mu)^q}{q}\sum_{n=0}^{\infty}(\mp i\mu)^{-n}\binom{q}{n}\left[\left(\frac{q}{2}\right)^n - \left(-\frac{q}{2}\right)^n\right].$$

Since in the sum only the odd $(n = 2k+1)$ terms contribute these two contributions add up giving an agreement with the gauge-theory correlator $G(J)$, under the replacement $-\mu^2 \to N^2$ with an overall factor $\pm i$. The sign is consistent with the correspondence of genus-expansion parameters we found at the planar level.

The fact that $|\mu| \sim N$ implies that the fermion levels must be filled evenly from the top of the potential for the ground state, quite differently from the usual double scaling limit in defining higher genus contributions in the traditional treatment of the $c = 1$ matrix model. This of course reflect the feature emphasized in the beginning of this subsection.

6. Conclusion

We have argued that the two-point correlators of 1/2-BPS multi-trace operators are interpreted as the S-matrix of the $c = 1$ Hermitian matrix model. The correspondence of both sides are fairly tight in the planar approximation. However, as we have cautioned, the situation is somewhat different if we consider the non-planar case, where we have presented evidence for the correspondence only in the limit of large momentum $J \to \infty$. To establish exact correspondence for finite J and large but finite N, we need a modified definition of the $c = 1$ model, in regard to the level spacings in filling fermion states from the top of the potential, not from the bottom, as we have alluded to in section 2. It must be defined such that the 'ground state' of the model becomes equivalent with that of the matrix model with positive harmonic potential and consequently we can have a one-to-one mapping even for excited states (again from top to down) to those of the positive harmonic potential.

We may also hope that, in the large J limit, we can discuss the genus corrections from the viewpoint of pp-wave string field theory. However, at the present state of development, even the question of what is the right string-field vertex for this purpose is not settled. For some recent works related to this question, see [23].

We have emphasized that unlike the known correspondence between $c = 1$ matrix model and the S-matrix of 2D string theory, the leg factor is not necessary. In ref. [8], it has been established that we need a special leg-like factor in order to relate the 3-point correlator of the BMN operator with the Euclidean S-matrix in the AdS bulk from boundary to boundary. However, for the extremal correlators such as

$$\langle \mathrm{Tr}(\bar{Z}^{J_1+J_2})(x)\mathrm{Tr}(Z^{J_1})(y)\mathrm{Tr}(Z^{J_2})(y')\rangle,$$

the corresponding 3-point (on-shell) vertices in the bulk vanish and simultaneously the leg factor diverges, so that the product of 3-point vertices and the leg factor gives finite correlators. In the limit $y \to y'$, these extremal 3-point correlators reduce smoothly to two-point correlators which are dealt with in the present work. Similar arguments apply also to higher-point extremal correlators. It is quite significant that the droplet picture provides us a direct correspondence for the correlators and Euclidean S-matrix in the bulk without such complications. It would be very interesting if there is any extension of our results to non-BPS operators.

Indeed, the c=1 interpretation might be of relevance to the full AdS string where analogous structures and $c = 1$ type scaling [24, 25] has been observed recently. The latter correspondence regarding the *world-sheet* S-matrix points towards capturing the mixing among excited but *single-body* string states. In contrast to this, in our case, the $c = 1$ matrix model is relevant in understanding the *multi-body* states of strings, restricted to the ground state of spin chain states.

We noted parallels with the relation between the so-called normal (complex) matrix model and the $c = 1$ model at finite temperature [18]. It would be of interest to clarify the connection further.

Finally, we mention also the recent investigation of [26] which discusses correlators and topology changes in the 1/2 BPS sector.

JHEP03(2007)001

Acknowledgments

One (T.Y.) of the present authors thanks the Physics Department of Brown University and the Center for Theoretical Physics, MIT for hospitality. Main part of the present work was done during T.Y.'s visit to both institutions.

The work of A. J. is supported in part by the Department of Energy under Contract DE-FG02-91ER40688,Task A .

The work of T. Y. is supported in part by Grant-in-Aid for Scientific Research (No. 13135205 (Priority Areas) and No. 16340067 (B)) from the Ministry of Education, Science and Culture, and also by Japan-US Bilateral Joint Research Projects from JSPS.

A. Genus expansion from exact formulas

Here we present some formulas which are useful for studying genus expansion on the basis of exact formulas for correlators. First we need an expansion formula [27] for the ratio of Gamma functions:

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha-\beta+1)}{n!\Gamma(\alpha-\beta-n+1)} B_n^{(\alpha-\beta+1)}(\alpha) z^{\alpha-\beta-n}, \tag{1.1}$$

where the coefficient function $B_n^{(\mu)}(x)$ is the generalized Bernoulli polynomial [28] expressed as

$$B_n^{(\mu)}(x) = \sum_{k=0}^{n} \frac{k!}{(2k)!} \binom{n}{k} \binom{\mu+k-1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{2k} (x+j)^{n-k}$$
$$\times F[k-n, k-\mu; 2k+1; \frac{j}{x+j}]. \tag{1.2}$$

In our case, the explicit expressions of the coefficients are

$$B_0^{(\alpha-\beta+1))}(\alpha) = 1,$$
$$B_1^{(\alpha-\beta+1)}(\alpha) = \frac{1}{2}(\alpha+\beta-1), \tag{1.3}$$
$$B_2^{(\alpha-\beta+1)}(\alpha) = \frac{1}{4}(\alpha+\beta)^2 - \frac{7}{12}\alpha - \frac{5}{12}\beta + \frac{1}{6}, \tag{1.4}$$
$$B_3^{(\alpha-\beta+1)}(\alpha) = \frac{1}{8}(\alpha+\beta)^3 - \frac{1}{2}\left(\alpha+\frac{\beta}{2}\right)(\alpha+\beta) + \frac{3}{8}\alpha + \frac{1}{8}\beta. \tag{1.5}$$

In particular for $\alpha-\beta \gg 1$, we have an asymptotic form

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum_{n=0}^{\infty} \binom{\alpha-\beta}{n} \frac{(\alpha+\beta)^n}{2^n} z^{\alpha-\beta-n}, \tag{1.6}$$

which can easily be understood by a 'dilute-gas' approximation for the power-series expansion of the above ratio in $1/z$. Note that in this limit $(\alpha+\beta)/2$ is the average value of the coefficients of $1/z$ in the product $\prod_{n=\beta}^{\alpha-1}(1+\frac{n}{z})$. This asymptotic formula is used in section 5.

Using the exact formulas [17] such as (4.7), (4.8) and, for $n = 3$,

$$\begin{aligned}\langle \mathrm{Tr}(\bar{Z}^J)\mathrm{Tr}(Z^{J_1})\mathrm{Tr}(Z^{J_2})\mathrm{Tr}(Z^{J_3})\rangle = \frac{1}{J+1}\Bigg(&\frac{\Gamma(N+J_1+J_2+J_3+1)}{\Gamma(N)} - \frac{\Gamma(N+J_2+J_3+1)}{\Gamma(N-J_1)} \\ &-\frac{\Gamma(N+J_1+J_3+1)}{\Gamma(N-J_2)} - \frac{\Gamma(N+J_1+J_2+1)}{\Gamma(N-J_3)} + \frac{\Gamma(N+J_1+1)}{\Gamma(N-J_2-J_3)} \\ &+\frac{\Gamma(N+J_2+1)}{\Gamma(N-J_1-J_3)} + \frac{\Gamma(N+J_3+1)}{\Gamma(N-J_1-J_2)} - \frac{\Gamma(N+1)}{\Gamma(N-J_1-J_2-J_3)}\Bigg) \quad \text{etc,}\end{aligned} \tag{1.7}$$

together with the above expansion formula (1.1), we can check the results of section 4, although the algebra becomes increasingly tedious for larger n.

References

[1] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [hep-th/0111222].

[2] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [hep-th/0403110].

[3] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174].

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[5] T. Yoneya, *Extended fermion representation of multi-charge 1/2-BPS operators in AdS/CFT: towards field theory of D-branes*, *JHEP* **12** (2005) 028 [hep-th/0510114].

[6] K. Okuyama, *1/2 BPS correlator and free fermion*, *JHEP* **01** (2006) 021 [hep-th/0511064].

[7] S. Dobashi, H. Shimada and T. Yoneya, *Holographic reformulation of string theory on $AdS_5 \times S^5$ background in the PP-wave limit*, *Nucl. Phys.* **B 665** (2003) 94 [hep-th/0209251].

[8] S. Dobashi and T. Yoneya, *Resolving the holography in the plane-wave limit of AdS/CFT correspondence*, *Nucl. Phys.* **B 711** (2005) 3 [hep-th/0406225]; *Impurity non-preserving 3-point correlators of BMN operators from PP-wave holography. I: bosonic excitations*, *Nucl. Phys.* **B 711** (2005) 54 [hep-th/0409058];
For further references, see a related recent work, A. Miwa and T. Yoneya, *Holography of Wilson-loop expectation values with local operator insertions*, *JHEP* **12** (2006) 060 [hep-th/0609007].

[9] L. Maoz and V.S. Rychkov, *Geometry quantization from supergravity: the case of 'bubbling AdS'*, *JHEP* **08** (2005) 096 [hep-th/0508059];
L. Maoz and V.S. Rychkov, *Geometry quantization from supergravity: the case of 'bubbling AdS'*, *JHEP* **08** (2005) 096 [hep-th/0508059].

[10] J. Polchinski, *Classical limit of (1+1)-dimensional string theory*, *Nucl. Phys.* **B 362** (1991) 125.

[11] G.W. Moore and R. Plesser, *Classical scattering in (1+1)-dimensional string theory*, *Phys. Rev.* **D 46** (1992) 1730 [hep-th/9203060].

[12] For a review, see A. Jevicki, *Development in 2-d string theory*, hep-th/9309115.

[13] K. Demeterfi, A. Jevicki and J.P. Rodrigues, *Scattering amplitudes and loop corrections in collective string field theory*, *Nucl. Phys.* **B 362** (1991) 173.

[14] E. Hsu and I.R. Klebanov, *Quantum corrections in collective field theory*, *Phys. Lett.* **B 321** (1994) 99 [hep-th/9310075].

[15] J. Lee, *Time dependent backgrounds in 2-d string theory and the S matrix generating functional*, *Phys. Rev.* **D 49** (1994) 2957 [hep-th/9310190].

[16] A. Jevicki, J.P. Rodrigues and A.J. van Tonder, *Scattering states and symmetries in the matrix model and two-dimensional string theory*, *Nucl. Phys.* **B 404** (1993) 91 [hep-th/9209057].

[17] N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *BMN correlators and operator mixing in $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 650** (2003) 125 [hep-th/0208178] and references therein;
another useful reference is M.C. Bergere, *Correlation functions of complex matrix models*, *J. Phys.* **A 39** (2006) 8749 [hep-th/0511019].

[18] S.Y. Alexandrov, V.A. Kazakov and I.K. Kostov, *2D string theory as normal matrix model*, *Nucl. Phys.* **B 667** (2003) 90 [hep-th/0302106].

[19] A. Mukherjee and S. Mukhi, *Noncritical string correlators, finite-N matrix models and the vortex condensate*, *JHEP* **07** (2006) 017 [hep-th/0602119].

[20] G.W. Moore, M.R. Plesser and S. Ramgoolam, *Exact S matrix for 2-D string theory*, *Nucl. Phys.* **B 377** (1992) 143 [hep-th/9111035].

[21] S.Y. Alexandrov, V.A. Kazakov and I.K. Kostov, *Time-dependent backgrounds of 2D string theory*, *Nucl. Phys.* **B 640** (2002) 119 [hep-th/0205079].

[22] M. O'Loughlin, *On consistent boundary conditions for $c = 1$ string theory*, *Nucl. Phys.* **B 449** (1995) 80 [hep-th/9501039].

[23] G. Grignani, M. Orselli, B. Ramadanovic, G.W. Semenoff and D. Young, *AdS/CFT vs. string loops*, *JHEP* **06** (2006) 040 [hep-th/0605080].

[24] C. Gómez and R. Hernández, *Integrability and non-perturbative effects in the AdS/CFT correspondence*, *Phys. Lett.* **B 644** (2007) 375 [hep-th/0611014].

[25] J. Maldacena and I. Swanson, *Connecting giant magnons to the PP-wave: an interpolating limit of $AdS_5 \times S^5$*, hep-th/0612079.

[26] T. Brown, R. de Mello Koch, S. Ramgoolam and N. Toumbas, *Correlators, probabilities and topologies in $N = 4$ SYM*, hep-th/0611290.

[27] F.G. Tricomi and A. Erdélyi, *The asymptotic expansion of a ratio of gamma functions*, *Pacific J. Math.* **1** (1951) 133.

[28] H.D. Strivastava and P.G. Todorov, *An explicit formula for the generalized Bernoulli polynomials*, *Jour. Math. Anal. Appl.* **130** (1988) 509.

JHEP03(2007)001